TURBULENT MHD FLOWS EXCITED BY ROTATING MAGNETIC FIELDS

A. Kapusta, B. Mikhailovich
Center for MHD Studies, Ben-Gurion University of the Negev, Beer-Sheva, Israel (borismic@bgu.ac.il))

At present, rotating magnetic fields (RMF) and traveling magnetic fields (TMF) are widely used in metallurgical technologies of the production of continuous ingots and castings, alloys of ferrous and non-ferrous metals, etc. Electrical parameters of electromagnetic stirring (EMS) are computed basing on technologically necessary values of mean velocity of the turbulent flow excited by EMS. Therefore, it is rather urgent to develop simple engineering methods of mean velocity computation.

A mathematical model of a two-dimensional mean turbulent flow of a viscous incompressible electrically conducting fluid excited by an amplitude-modulated two-dimensional RMF in vessels of circular and rectangular cross-sections is formulated in an induction-free approximation using a semi-empirical model of "external" friction. A two-dimensional RMF is excited by an ideal inductor with one pair of poles, which constitutes a cylindrical cavity of circular cross-section with the radius R_0 cut out in an ideal ferromagnetic ($\mu \to \infty$), with an infinitely thin current layer traveling over its surface.

Electrodynamic processes in a conducting medium contained in the inductor bore are described in a cylindrical coordinate system r, ϕ, z by the following dimensionless equation for the z-component of the magnetic induction vector potential ($\mathbf{b} = \operatorname{rot} \mathbf{a}$):

$$\frac{\varpi_0}{2\pi}\left(\frac{\partial a_z}{\partial \tau} + \frac{\bar{v}_\varphi}{\bar{r}}\frac{\partial a_z}{\partial \bar{\varphi}}\right) = \Delta a_z \, , \tag{1}$$

where $\varpi_0 = \mu_0 \sigma \omega_0 R_0^2$; $\tau = \omega_0 t/2\pi$; $\bar{r} = r/R_0$; $\bar{\varphi} = \varphi/2\pi$; $a_z = A_z/A_0$; $A_0 = \mu_0 NIR_0$; $\bar{v}_\varphi = V_\varphi/\omega R_0$; $\Delta = \dfrac{\partial^2}{\partial \bar{r}^2} + \dfrac{1}{\bar{r}}\dfrac{\partial}{\partial \bar{r}} + \dfrac{1}{4\pi^2 \bar{r}^2}\dfrac{\partial^2}{\partial \bar{\varphi}^2}$; NI is a linear current load of the inductor (hereinafter all the variables are not overlined). Eq. (1) is solved with the boundary condition

$$\left.\frac{\partial a_z}{\partial r}\right|_{r=1} = -\left[1 + e \sin 2\pi \left(k\tau - \varphi + \beta\right)\right] \sin 2\pi \left(\tau - \varphi\right) , \tag{2}$$

where e is the dimensionless depth of amplitude modulation, β is the phase shift between the carrier and modulating waves,$2\pi k$ is a dimensionless modulation frequency.

In the approximation of small $\varpi < 1$, the solution of problems (1), (2) is:

$$a_z = -\left[r \sin 2\pi\theta_0 - e\, r^2 \left(\cos 4\pi\theta_2 + \cos 4\pi\theta_1\right)/4\right] \, , \tag{3}$$

where $\theta_0 = \tau - \varphi$; $\theta_1 = (k-1)\tau + \beta$; $\theta_2 = (k+1)\tau/2 - \varphi + \beta/2$.

Using the known relationships between the vectorial potential, current density and r- and ϕ-components of the magnetic induction, we obtain the following expression:

$$\begin{aligned}\mathrm{rot}_z\,\mathbf{f} = \frac{1}{r}\frac{\partial\,(rf_\varphi)}{\partial r} - \frac{1}{2\pi r}\frac{\partial f_r}{\partial \varphi} = \frac{\varpi_0}{2}\Big[2S_1 + e^2S_2r^2 + \frac{5er}{2}S_1\sin 2\pi\theta_3 - \\ - \frac{er}{2}S_6\sin 2\pi\theta_5 - \frac{e^2}{2}r^2S_8\cos 4\pi\theta_1 + \frac{e^2}{4}r^2S_7\cos 4\pi\theta_3\Big]\,,\end{aligned} \tag{4}$$

where $S_6 = 2k - 3 + \Omega$; $S_7 = 2k - 1 - \Omega$; $S_8 = k - 2 + \Omega$.

We write (4) in the form

$$\mathrm{rot}_z\, f = \varpi_0\big[f_0(x,y) + e\cdot f_1(x,y,\tau)/2\big]/2. \tag{5}$$

In a general case, using the model of "external" friction, a flow function of the non-stationary MHD flow is described by the following dimensionless equation in the Cartesian system of coordinates x, y, z:

$$\frac{1}{2\pi}\frac{\partial\Delta\psi_z}{\partial\tau} = \frac{1}{\mathrm{Re}_\omega}\Delta^2\psi_z + \lambda^*\Delta\psi_z - N_\omega f(x,y,\tau) \tag{6}$$

with boundary conditions on the contour Γ:

$$\left.\frac{\partial\psi_z}{\partial n}\right|_r = \psi_z|_r = 0 \tag{7}$$

and periodicity condition

$$\psi_z(\tau) = \psi_z(\tau+1)\,, \tag{8}$$

where $\mathrm{Re}_\omega = \omega_0 R_H^2/\nu$, $N_\omega = \mathrm{Ha}_a^2/\mathrm{Re}_\omega$, $\mathrm{Ha}_a = B_aR_H\sqrt{\sigma/\eta}$, $R_H = 2ab/(a+b)$, $\lambda^* = C_\varepsilon \mathrm{Re}_\omega^{-\varepsilon}\langle\Omega\rangle^{1-\varepsilon}/\delta_z$, $C_\varepsilon = 0.0184\cdot e^{11.82\varepsilon}$, $\langle\Omega\rangle = \dfrac{1}{8ab}\displaystyle\int_{-a}^{a}\int_{-b}^{b}|\Delta\psi_z|\,\mathrm{d}\,x\mathrm{d}y$, ε is a structural constant representing the only empirical parameter of the mathematical model, whose value determines the "external" friction coefficient and the dynamic characteristics of the mean turbulent flow.

Using the assumption $\mathrm{Re}_\omega^{-1} \ll \lambda^*$, we obtain:

$$\frac{1}{2\pi}\frac{\partial\Delta\psi_z}{\partial\tau} - \lambda^*\Delta\psi_z = -N_\omega\left[f_0(x,y) + \frac{e}{2}f_1(x,y,\tau)\right] \tag{9}$$

$$\psi_z|_\Gamma = 0; \qquad \psi_z(\tau) = \psi_z(\tau+1) \tag{10}$$

We seek the solution of the quasi-linear problem (9), (10) as a sum of a stationary and non-stationary solutions:

$$\psi_z = \Psi_0(x,y) + \Psi_1(x,y,\tau). \tag{11}$$

The stationary problem

$$\Delta\psi_0 = \frac{N_\omega}{\lambda^*}f_0(x,y),\ \psi_0|_\Gamma = 0\,, \tag{12}$$

where $f_0(x,y) = 2S_1 + e^2S_2\left(x^2+y^2\right)$, has a simple solution:

$$\begin{aligned}\psi_0 = \sum_{n=1}^{\infty}\left(Q_{n1}+Q_{n2}\right)\Big(\frac{\mathrm{ch}\nu_n x}{\mathrm{ch}\nu_n a} - 1\Big)\cos\nu_n y + \\ + \left(Q_{n1}^*+Q_{n2}^*\right)\Big(\frac{\mathrm{ch}\mu_n y}{\mathrm{ch}\mu_n b} - 1\Big)\cos\mu_n x\,,\end{aligned} \tag{13}$$

where $Q_{n1} = \dfrac{2N_\omega S_1 \cdot (-1)^n}{\lambda^* b\nu_n^3}$; $Q^*_{n1} = \dfrac{2N_\omega S_1 \cdot (-1)^n}{\lambda^* a\mu_n^3}$; $Q_{n2} = \dfrac{2N_\omega S_2 e^2 \cdot (-1)^n}{\lambda * b\nu_n^5}$;

$$Q^*_{n2} = \frac{2N_\omega S_2 e^2 \cdot (-1)^n}{\lambda * a\mu_n^5}; \quad \nu_n = \pi\Big(n - \frac{1}{2}\Big)/b; \quad \mu_n = \pi\Big(n - \frac{1}{2}\Big)/a.$$

The non-stationary problem has the form

$$\frac{1}{2\pi}\frac{\partial \Delta\psi_1}{\partial \tau} - \lambda^* \Delta\psi_1 = -N_\omega \frac{e}{2} f_1(x,y,\tau) \tag{14}$$

$$\psi_1|_\Gamma = 0; \qquad \psi_1(\tau) = \psi_1(\tau+1) \tag{15}$$

where $f_1(x,y,\tau) = 5S_1\sqrt{x^2+y^2}\sin 2\pi\theta_3 - S_6\sqrt{x^2+y^2}\sin 2\pi\theta_5 -$

$$-S_8\left(x^2+y^2\right)\cos 4\pi\theta_1 + S_7\left(x^2+y^2\right)\cos 4\pi\theta_3.$$

Since the arguments of trigonometric functions contain a variable $\phi = \mathrm{arctg}\,(\mathrm{y/x})$, we can rewrite: $f_1(x,y,\tau) = \sum\limits_{m=1}^{4}\left\{F_m(x,y)\sin\omega_m\tau + \Phi_m(x,y)\cos\omega_m\tau\right\}$, where

m	ω_m	F_m	Φ_m
1	$2\pi k$	$5S_1x$	$5S_1y$
2	$2\pi(k-2)$	S_6x	$-S_6y$
3	4π	$-S_8xy\sqrt{x^2+y^2}$	$-S_8\big(x^2-y^2\big)\sqrt{x^2+y^2}$
4	$4\pi k$	$-S_7\big(x^2-y^2\big)\sqrt{x^2+y^2}$	$S_7xy\sqrt{x^2+y^2}$

The solution of (14), (15) has the form:

$$\psi_1 = \sum_{m=1}^{4}\sum_{i,j=1}^{\infty} S_{1mij}(x,y)\sin\omega_m\tau + S_{2mij}(x,y)\cos\omega_m\tau, \tag{16}$$

where $S_{1mij}(x,y) = \psi_{1Smij}\cos\mu_i x\cdot\cos\nu_j y + \psi^*_{1Smij}\sin\dfrac{i\pi}{a}x\cdot\sin\dfrac{j\pi}{b}y$;

$$S_{2mij}(x,y) = \psi_{1Cmij}\cos\mu_i x\cdot\cos\nu_j y + \psi^*_{1Cmij}\sin\frac{i\pi}{a}x\cdot\sin\frac{j\pi}{b}y\,;$$

$$\psi_{1Smij} = \frac{\pi N_\omega e\int\limits_0^a\int\limits_0^b (F_m\lambda^* - \Phi_m\omega_m)\cos\mu_i x\cdot\cos\nu_j y\cdot \mathrm{d}x\,\mathrm{d}y}{(\lambda^{*2}+\omega_m^2)(\mu_i^2+\nu_j^2)ab};$$

$$\psi^*_{1Smij} = \frac{\pi N_\omega e\int\limits_0^a\int\limits_0^b (F_m\lambda^* - \Phi_m\omega_m)\sin\frac{i\pi}{a}x\cdot\sin\frac{j\pi}{b}y\cdot \mathrm{d}x\,\mathrm{d}y}{\pi^2(\lambda^{*2}+\omega_m^2)\left[(i/a)^2+(j/b)^2\right]ab}.$$

Expressions for ψ_{1Cmij}, ψ^*_{1Cmij} can be obtained by substituting $(F_m\lambda^* - \Phi_m\omega_m)$ in the integrands with $(F_m\omega_m + \Phi_m\lambda^*)$.

To compute the value of Ω appearing in the zero approximation of λ^*, the following equation is used:

$$\Omega_0^{2-\varepsilon} + Q_\varepsilon\left[1+e^2(a^2+b^2)/6\right]\Omega_0 - Q_\varepsilon\left[1+e^2(k+1)(a^2+b^2)/12\right] = 0\,, \tag{17}$$

where $Q_\varepsilon = \mathrm{Ha}^2\cdot\delta_z/\mathrm{Re}_\omega^{1-\varepsilon}\cdot C_\varepsilon$.

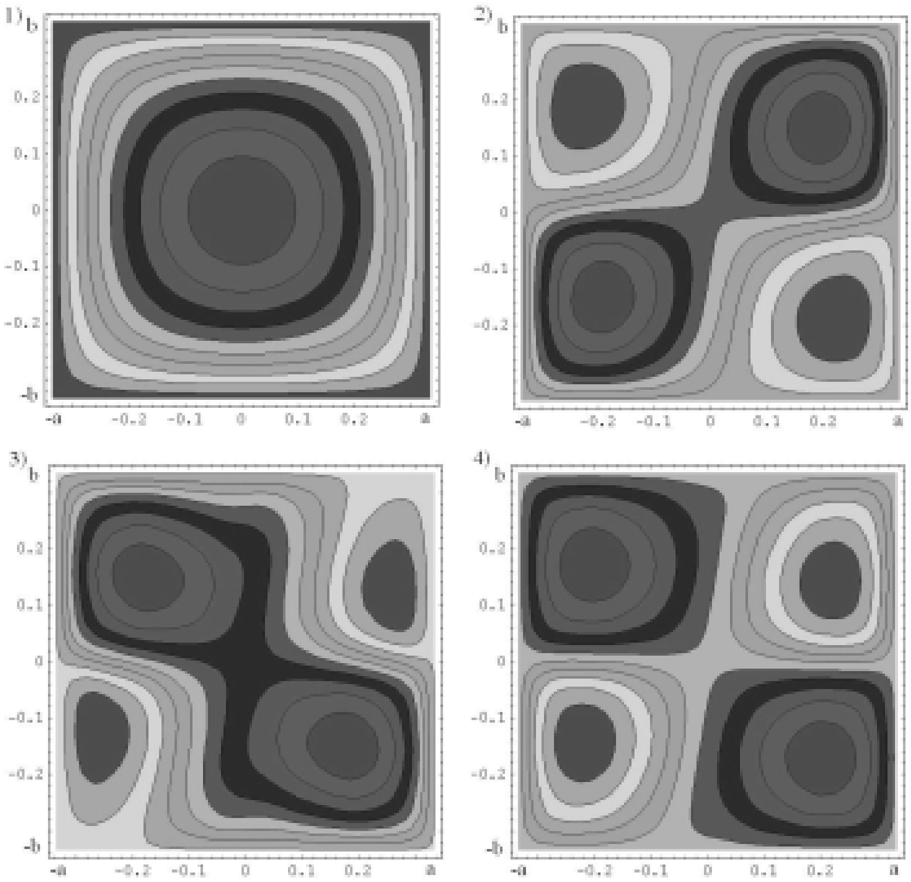

Fig. 1.

Contours of stationary and non-stationary flows at different moments of time computed using Eqs. (13), (16) are presented in Fig. 1.