MIGRATION OF A SOLID CONDUCTING SPHERE IMMERSED IN A LIQUID METAL NEAR A PLANE BOUNDARY UNDER THE ACTION OF UNIFORM AMBIENT ELECTRIC AND MAGNETIC FIELDS

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Introduction. As established in [1], a solid and conducting sphere with radius a and conductivity $\sigma_s \geq 0$ freely suspended in a Newtonian liquid metal of uniform viscosity μ, and conductivity $\sigma \geq 0$ and subject to uniform ambient electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ translates without rotating and parallel to $\mathbf{E} \wedge \mathbf{B}$ at the velocity $\mathbf{U}$ such that

$$\mathbf{U} = a^2(\sigma_s - \sigma)\frac{\mathbf{E} \wedge \mathbf{B}}{3\mu(\sigma_s + 2\sigma)}. \tag{1}$$

This work examines, within the same framework, the rigid-body motion (translation and rotation) of a sphere when it lies near a plane solid wall.

1. Governing problem and symmetries. We consider, as sketched in Fig. 1, a solid conducting sphere with uniform conductivity $\sigma_s \geq 0$, radius a and center O', held fixed in a Newtonian liquid metal of uniform viscosity μ and conductivity $\sigma \geq 0$ above a rigid and stationary plane wall Σ. Cartesian coordinates (O, x_1, x_2, x_3) are used with Σ the $x_3 = 0$ plane, $\mathbf{OO'} = l\mathbf{e}_z$ and $l > a$. We look at the net magnetohydrodynamic force $\mathbf{F}_n$ and torque $\mathbf{C}_n$ (about O') exerted on the sphere when subject to uniform electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. The wall is perfectly insulating or conducting for $\mathbf{E}$ respectively parallel to or normal to $\mathbf{e}_3$ whilst the disturbed electric field is $\mathbf{E} - \nabla\phi'$ in the sphere $\mathcal{P}$ and $\mathbf{E} - \nabla\phi$ in the liquid domain Ω. Setting $\mathbf{n} = \mathbf{O'M}/a$ on the sphere's surface S, the functions ϕ and ϕ' obey

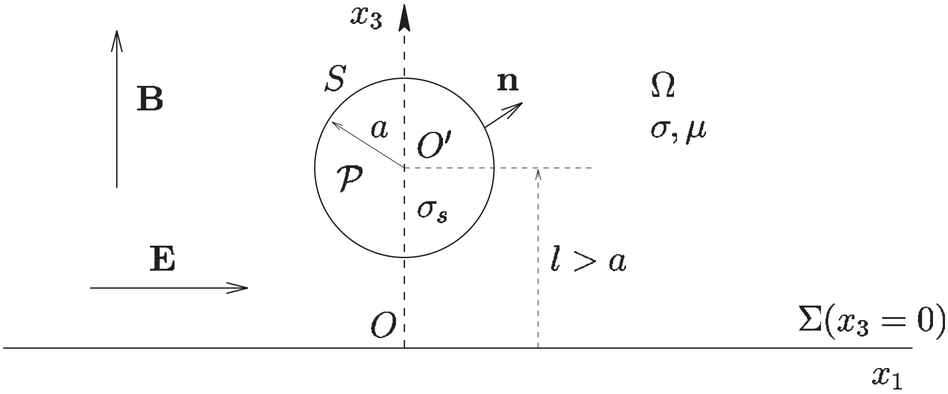

Fig. 1. A conducting solid sphere held fixed or fixed or freely-suspended above the palne wall Σ in a Newtonain liquid metal and subject to uniform electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

Table 1. Relevant Cases m ($m = 1, \ldots, 5$) and associated non-zero Cartesian components of the net force $\mathbf{F}$ and torque $\mathbf{C}$ on a *motionless* sphere and of the trnalsational velocity $\mathbf{U}$ and angular velocity $\mathbf{\Omega}$ of a *freely suspended* sphere.

Case m	wall type	$\mathbf{E}$	$\mathbf{B}$	$\mathbf{F}_\mathrm{n}$	$\mathbf{C}_\mathrm{n}$	$\mathbf{U}$	$\mathbf{\Omega}$
1	insulating	$E\mathbf{e}_2$	$B\mathbf{e}_3$	$F_\mathrm{n}^{(1)}\mathbf{e}_1$	$C_\mathrm{n}^{(1)}\mathbf{e}_2$	$U^{(1)}\mathbf{e}_1$	$\Omega^{(1)}\mathbf{e}_2$
2	insulating	$E\mathbf{e}_2$	$B\mathbf{e}_1$	$F_\mathrm{n}^{(2)}\mathbf{e}_3$	$\mathbf{0}$	$U^{(2)}\mathbf{e}_3$	$\mathbf{0}$
3	insulating	$E\mathbf{e}_2$	$B\mathbf{e}_2$	$\mathbf{0}$	$C_\mathrm{n}^{(3)}\mathbf{e}_3$	$\mathbf{0}$	$\Omega^{(3)}\mathbf{e}_3$
4	conducting	$E\mathbf{e}_3$	$B\mathbf{e}_3$	$\mathbf{0}$	$C_\mathrm{n}^{(4)}\mathbf{e}_3$	$\mathbf{0}$	$\Omega^{(4)}\mathbf{e}_3$
5	conducting	$E\mathbf{e}_3$	$B\mathbf{e}_2$	$F_\mathrm{n}^{(5)}\mathbf{e}_1$	$C_\mathrm{n}^{(5)}\mathbf{e}_2$	$U^{(5)}\mathbf{e}_1$	$\Omega^{(5)}\mathbf{e}_2$

$$\nabla^2\phi' = 0 \quad \text{in} \quad \mathcal{P}, \quad \nabla^2\phi = 0 \quad \text{in} \quad \Omega, \quad \nabla\phi = 0 \quad \text{as} \quad OM \to \infty, \tag{2}$$

$$\sigma(\mathbf{E} - \nabla\phi)\cdot\mathbf{n} \quad \text{and} \quad \phi = \phi' \quad \text{on} \quad S, \tag{3}$$

$$\nabla\phi\cdot\mathbf{e}_3 = 0 \quad \text{on} \quad \Sigma \quad \text{if} \quad \mathbf{E}\cdot\mathbf{e}_3 = 0, \quad \phi = 0 \quad \text{on} \quad \Sigma \quad \text{if} \quad \mathbf{E}\wedge\mathbf{e}_3 = 0. \tag{4}$$

The liquid flows with pressure p, velocity $\mathbf{u}$ and stress tensor $\boldsymbol{\sigma}$ because of the Lorentz body force $\mathbf{f} = \sigma(\mathbf{E}\nabla\phi + \mathbf{u}\wedge\mathbf{B})$ where one assumes that $\mathbf{B}$ is not disturbed [1]. Accordingly, one obtains

$$\mathbf{F}_\mathrm{n} = \mathbf{F}_i + \mathbf{F}, \quad \frac{1}{\sigma_\mathrm{s}}\mathbf{F}_i = \int_{\mathcal{P}} (\mathbf{E} - \nabla\phi)\wedge\mathbf{B}\,\mathrm{d}\Omega, \quad \mathbf{F} = \int_S \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}S, \tag{5}$$

$$\mathbf{C}_\mathrm{n} = \mathbf{C}_i + \mathbf{C}, \quad \frac{1}{\sigma_\mathrm{s}}\mathbf{C}_i = \int_{\mathcal{P}} \mathbf{O'M}\wedge[(\mathbf{E} - \nabla\phi)\wedge\mathbf{B}]\,\mathrm{d}\Omega, \quad \mathbf{C} = \int_S \mathbf{O'M}\wedge\boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}S. \tag{6}$$

Assuming vanishing Reynolds and Hartmann numbers [2], $(\mathbf{u}, p)$ satisfies

$$\nabla\cdot\mathbf{u} = 0 \quad \text{and} \quad \mu\nabla^2\mathbf{u} = \nabla p - \sigma(\mathbf{E} - \nabla\phi)\wedge\mathbf{B} \quad \text{in} \quad \Omega, \tag{7}$$

$$\mathbf{u} = 0 \;\text{ on }\; S, \quad \mathbf{u} = 0 \;\text{ on }\; \Sigma, \quad (\mathbf{u}, p) \to (\mathbf{0}, \sigma[\mathbf{E}\wedge\mathbf{B}]\cdot\mathbf{O'M}) \quad \text{as} \quad O'M \to \infty. \tag{8}$$

By linearity and for symmetry reasons it is possible to restrict the analysis to five Cases m ($m = 1, \ldots, 5$) defined in the Table 1. Furthermore, exploiting symmetry considerations as in [2] permits us to obtain for these Cases the direction of $\mathbf{F}$, $\mathbf{C}$ for the *motionless* sphere and of the translational velocity $\mathbf{U}$ and angular velocity $\mathbf{\Omega}$ of a *freely suspended* sphere. The results, summarized in the Table 1, show that each pair $(\mathbf{F}, \mathbf{C})$ and $(\mathbf{U}, \mathbf{\Omega})$ solely depends upon 7 unknown coefficients for a general setting $(\mathbf{E}, \mathbf{B})$.

2. Advocated coordinates and flow decomposition. By virtue of (5)–(6), one gets the net force $\mathbf{F}$ and net torque $\mathbf{C}$ on the motionless sphere by successively evaluating the pairs $(\mathbf{F}_i, \mathbf{C}_i)$ and $(\mathbf{F}, \mathbf{C})$. This task is achieved as detailed below.

2.1. Evaluation of $((\mathbf{F}_i, \mathbf{C}_i)$. The vectors $\mathbf{F}_i$ and $\mathbf{C}_i$ are obtained by solving the problem (2)–(4). The fluid domain's geometry suggests to use for this purpose the suitable bipolar coodinates (ξ, η, ψ) which relate [3–4] to the usual cylindrical polar coordinates ((ρ, x_3, ψ), with $x_1 = \rho\cos\psi$ and $x_2 = \rho\sin\psi$, as follows

$$\rho = \frac{c\sinh\xi}{\cosh\xi - \cos\eta}, x_3 = \frac{c\sin\eta}{\cosh\xi - \cos\eta}, c = (l^2 - a^2)^{1/2}. \tag{9}$$

Under this choice, the surfaces S and Σ admit the equation $\xi = a$ and $\xi = 0$, respectively with $l = a\cosh\alpha$. Similary to the treatment available in [5] it is then possible to expand each non-zero Cartesian component of $\mathbf{F}_i$ and $\mathbf{C}_i$ as a serie of known coefficients that solely depend upon $(\alpha, a, \sigma_\mathrm{s}, \sigma)$ and (E, B) for each Case m.

2.2. Flow decomposition and evaluation of $(\mathbf{F}, \mathbf{C})$. On order to get ride of the body force arising in (7) it is fruitful to set $\mathbf{u} = \mathbf{u}_i + \mathbf{u}_2$ and $p = p_1 + p_2$ with $\mathbf{u}_1 = \sigma\phi(\mathbf{O'M} \wedge \mathbf{B})/(2\mu)$ and $p_1 = \sigma(\mathbf{E} \wedge \mathbf{B}) \cdot (\mathbf{O'M})$. As the reader may easily check, one thus arrives for the flow $(\mathbf{u}_2, p_2)$ at the problem

$$\nabla \cdot \mathbf{u}_2 = \nabla \cdot \mathbf{u}_1 \quad \text{and} \quad \mu\nabla^2\mathbf{u}_2 = \nabla p_2 \quad \text{in} \quad \Omega, \tag{10}$$

$$\mathbf{u}_2 = \mathbf{u}_1 \text{ on } S, \quad \mathbf{u}_2 = \mathbf{u}_1 \text{ on } \Sigma, \quad (\mathbf{u}_2, p_2) \to (\mathbf{0}, 0) \quad \text{as} \quad O'M \to \infty. \tag{11}$$

Indeed, the velocity $\mathbf{u}_2$ vanishes far from the sphere because so do $\mathbf{u}$, $\nabla\phi$ (and thus $\mathbf{u}_1$). Note that $(\mathbf{u}_2, p_2)$ is free from body force. We denote by $\boldsymbol{\sigma}_l$ the stress tensor associated to the flow $(\mathbf{u}_l, p_l)$ and note that $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$ with the definitions

$$\mathbf{F}_l = \int_S \boldsymbol{\sigma}_l \cdot \mathbf{n}\, \mathrm{d}S, \quad \mathbf{C}_l = \int_S \mathbf{O'M} \wedge \boldsymbol{\sigma}_l \cdot \mathbf{n}\, \mathrm{d}S, \quad \text{for} \quad l = 1, 2. \tag{12}$$

The simple form adopted by the flow $(\mathbf{u}_1, p_1)$ easily yields *on* S the basic relation

$$\boldsymbol{\sigma}_1 \cdot \mathbf{n} = a\sigma[\nabla\phi \cdot \mathbf{n}][\mathbf{n} \wedge \mathbf{B}]/2 - \sigma[(\mathbf{E} \wedge \mathbf{B}) \cdot \mathbf{O'M}]\mathbf{n} \tag{13}$$

which thus permits one to deduce from the previous determination of ϕ on the sphere's surface the pair $(\mathbf{F}_1, \mathbf{C}_1)$. Finally, the pair $(\mathbf{F}_2, \mathbf{C}_2)$ is obtained by solving (10)–(11) in bipolar coordinates. Such a tricky task is achieved by extending the treatment employed in [6–8] for divergence-free Stokes flows, about a solid translating or rotating sphere, that vanish on the wall.

3. Solution for Case 4. For conciseness, it is not possible to produce here the results for each Case m. We thus illustrate the method for the simple Case 4 and postpone the treatment of other Cases to the oral presentation.

4. Form of the potential ϕ in the liquid and value of $(\mathbf{F}_2, \mathbf{C}_2)$. Since $\mathbf{E} = E\mathbf{e}_3$ and $\mathbf{B} = B\mathbf{e}_3$ one arrives in the liquid, i. e. for $\xi \geq \alpha$, at

$$\phi = Ec(\cosh\xi - \lambda)^{1/2} \sum_{n\geq 0} B_n \sinh(\gamma_n\xi) P_n(\lambda) \tag{14}$$

with $\gamma_n = n+1/2$, $\lambda = \cos\eta$ and P_n the Legendre polynomial of order n. Moreover, setting $\delta = \sigma_\mathrm{s}/\sigma$, the coefficients B_n obey the linear system

$$\begin{aligned} n[\delta \sinh(\gamma_n 1)\alpha + \cosh(\gamma_n 1)\alpha]B_{n-1} + \\ + (1\delta)\sinh\alpha \sinh\gamma_n\alpha + (2n+1)\cosh\alpha[\cosh\gamma_n\alpha + \delta\sinh\gamma_n\alpha]B_n - \\ - (n+l)[\cosh(\gamma_n+1)\alpha + \delta\sinh(\gamma_n+1)\alpha]B_{n+1} = \\ = 2(1-\delta)\sqrt{2}\mathrm{e}^{-\gamma_n\alpha}[\cosh\alpha - (2n+1)\sinh\alpha] \text{ for } n \geq 0. \end{aligned} \tag{15}$$

By elementary algebra one thus establishes that $\mathbf{F}_i = 0$ and $\mathbf{C}_i = C_i^{(4)}\mathbf{e}_3$ with

$$C_i^{(4)} = -8\pi a^4 \sigma_\mathrm{s} EB \sinh^2\alpha \sum_{n\geq 0} B_n \sinh(\gamma_n\alpha) T_n, \tag{16}$$

$$T_0 = v_1, \quad (2n+1)T_n = v_{n+1} - v_{n-1} \text{ for } n \geq 1, \tag{17}$$

$$v_n = \sqrt{2}(n+1)\mathrm{e}^{-\gamma_n\alpha}[(2n+1)\sinh\alpha + 2\cosh\alpha]/15 \text{ for } n \geq 0. \tag{18}$$

4.1. Determination of $(\mathbf{F}_1, \mathbf{C}_1)$ *and* $(\mathbf{F}_2, \mathbf{C}_2)$. Using (13) in conjunction with (14) yields ($\mathbf{F}_1 = 0$ and $\mathbf{C}_1 = C_1^{(4)}\mathbf{e}_3$ with the following value

$$C_1^{(4)} = -4\pi a^4 \sigma EB \sinh^2\alpha \sum_{n\geq 0} B_n c_n(\alpha) v_n, \tag{19}$$

$$2c_n(\alpha) = [\sinh\alpha\sinh(\gamma_n\alpha) + (2n+1)\cosh\alpha\cosh(\gamma_n\alpha)]B_n - \\ - n[\cosh(\gamma_n - 1)\alpha]B_{n-1} - (n+1)\cosh(\gamma_n+1)\alpha]B_{n+1} \text{ for } n \geq 0. \tag{20}$$

Note that $\mathbf{u}_1$ vanishes on the plane wall Σ whereas $\nabla\cdot\mathbf{u}_1 = 0$ in the whole liquid domain. The problem (10)-(11) then becomes simple and symmetries suggest to select its solution as $p_2 = 0$ and $\mathbf{u}_2 = \sigma BF(\rho, x_3)\mathbf{e}_\psi/(2\mu)$ with $\mathbf{e}_\psi = \mathbf{e}_3 \wedge (\mathbf{e}_1 + \mathbf{e}_2)/(x_1^2 + x_2^2)$ for $\rho \neq 0$. Proceeding as in [9], one gets $\mathbf{F}_2 = \mathbf{0}$) and $\mathbf{C}_2 = C_2^{(4)}\mathbf{e}_3$ with

$$C_2^{(4)} = -2\sqrt{2}\pi a^4 \sigma EB \sinh^4\alpha \sum_{n\geq 1} n(n+1)G_n, \tag{21}$$

$$-\frac{n-1}{2n-1}\sinh(\gamma_{n-1}\alpha)G_{n-1} + \cosh\alpha\sinh(\gamma_n\alpha)G_n - \frac{n+2}{2n+3}\sinh(\gamma_{n+1}\alpha)G_{n+1} = \\ = \frac{\sinh(\gamma_{n-1}\alpha)}{2n-1}B_{n-1} - \frac{\sinh(\gamma_{n+1}\alpha)}{2n+3}B_{n+1} \text{ for } n \geq 1. \tag{22}$$

In summary, one computes $C_n^{(4)} = C_i^{(4)} + C_1^{(4)} + C_2^{(4)}$ by solving the systems (15), (22) and using the results (16)–(18), (19)–(20) and (21).

5. Concluding remarks. The oral presentation will not only give the net force $\mathbf{F}$ and net torque $\mathbf{C}$ applied on a *motionless* sphere in other Cases m but also obtain the rigid-body motion $(\mathbf{U}, \mathbf{\Omega})$ of a freely suspended sphere in each Case. Gravity effects with a uniform gravity field $g\mathbf{e}_3$ normal to the wall will be also added in Case 2 with a special attention to the possible equilibrium positions of the sphere versus (E, B, g, δ, d_s) with d_s) the sphere density with respect to the liquid metal.

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