EDGE INSTABILITY OF A LIQUID METAL SHEET IN A TRANSVERSE HIGH-FREQUENCY AC MAGNETIC FIELD

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1. Introduction. In several induction melting process, like the cold crucible orsemi-levitation, the free surface of liquid metal is submitted to AC magnetic fields which may considerably deform the surface. It has been observed that the free surface sometimes becomes strongly asymmetric and even irregular when too strong magnetic field is applied [1, 2]. On the one hand, such kind of instability lies outside the scope of the simple theoretical models considering a flat surface along a uniform AC magnetic field [3, 4]. On the other hand, non-planar surfaces in nonhomogenous magnetic fields do not admit, in general, an analytical solution. In this work, we propose a new, simple theoretical model to describe such instabilities. The model consists of a flat liquid metal droplet in a transverse AC magnetic field. The AC frequency is assumed to be so high that the magnetic field is effectively expelled from the droplet by the skin effect. On the other hand, the droplet is assumed to be thin so that it can be considered as a liquid sheet.

2. The model of a semi-infinite thin liquid sheet. Consider a thin horizontal layer of liquid metal submitted to a transverse AC magnetic field with induction $\vec{B}$. The layer is assumed to be semi-infinite and lay in the right-hand side of the $x-z$-plane so that the unperturbed edge of the layer coincides with the z-axis and the magnetic field is applied along the y-axis of the Cartesian system of coordinates, as illustrated in Fig. 1. The AC frequency is assumed to be so high that the skin effect renders the layer effectively impermeable to the magnetic field. In addition, the layer is assumed to be thin so that it can be considered as a thin sheet. The scalar magnetic potential Ψ, which is introduced to define the magnetic field in the space around the sheet as $\mathbf{B} = \boldsymbol{\nabla}\Psi$, satisfies $\nabla^2\Psi = 0$. The impermeability condition at both sides of the sheet takes the form

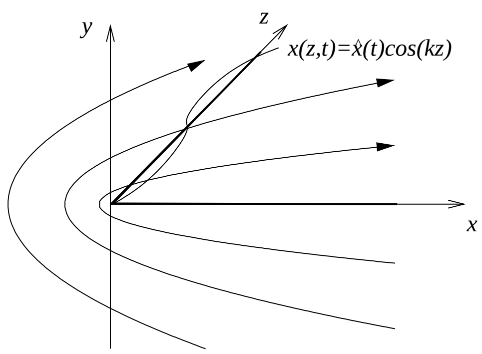

Fig. 1. Sketch to the formulation of the problem.

$$(\mathbf{n}\cdot\mathbf{B})|_{y=\pm0;\,x>0} = \left.\frac{\partial\Psi}{\partial n}\right|_{y=\pm0;\,x>0} = 0, \tag{1}$$

where $\mathbf{n}$ is the surface normal vector. In the following, we focus on the distribution of the magnetic field in the vicinity of the edge which can conveniently be described in the cylindrical coordinates with the z-axis coinciding with the edge, while the polar angle φ is measured as usual counterclockwise from the x-axis (see Fig.2a). The solution for the unperturbed potential in the vicinity of the edge satisfying condition (1) can be found as $\Psi_0(r,\varphi) = f_0(r)\cos(\varphi/2)$, where $f_0(r) = C_0\sqrt{r}$ involving C_0 an unknown constant, which can be determined by considering a strip of finite width. Further, suppose that there is a perturbation of the edge

$$x_1(z,t) = \hat{x}(t)\cos(kz)$$

with a small, generally time-dependent amplitude $\hat{x}(t)$ and the wave number k along the z-axis. The edge perturbation is expected to cause perturbation of the potential which can be presented as $\Psi(r,\varphi,z) = \Psi_0(r,\varphi) + \varepsilon\Psi_1(r,\varphi,z) + \ldots$, where Ψ_1 is a perturbation of the potential with a characteristic amplitude ε. To relate the perturbation of the potential to that of the edge, we need an additional condition at the edge, which follows from the induced current in the sheet. As easy to see, the current in the sheet and the magnetic field along it are mutually perpendicular because of $\mathbf{j} = \mu_0\mathbf{n}\times\mathbf{B}$, where μ_0 is the permeability of vacuum. Consequently, the magnetic field along the sheet has be to perpendicular to the edge because the current has to flow along the latter. Thus, along the edge L, we have $(\tau\cdot\mathbf{B})|_L = \partial\Psi/\partial\tau|_L = 0$, where τ is the unit vector tangential to the edge that implies $\Psi|_L = \text{const}$, where we can set const $= 0$ because the potential is defined up to an additive constant. By applying this condition at the perturbed edge with $x = x_1(z,t)$ we obtain up the first order terms in the perturbation amplitude $\Psi|_{x=x_1} \approx (\Psi_0 + x_1\partial\Psi_0/\partial x + \varepsilon\Psi_1)|_{r\to0;\,\varphi=0} = 0$ that results in

$$\varepsilon\Psi_1|_{r\to0;\,\varphi=0} = -\left.\frac{\partial\Psi_0}{\partial x}x_1\right|_{r\to0;\,\varphi=0} = -\left.\frac{C_0}{2}\frac{x_1}{\sqrt{r}}\right|_{r\to0}. \tag{2}$$

The perturbation of the potential satisfying the impermeability condition can be sought similarly to the base field in the form $\hat{\Psi}_1(r,\varphi) = f_1(r)\cos(\varphi/2)$ that leads to $\frac{1}{r}\frac{d}{dr}\left(r\frac{df_1}{dr}\right) + \frac{1}{4}\frac{f_1}{r^2} - kf_1 = 0$. The solution of this equation satisfying condition (2) is $f_1(r) = -\frac{C_0}{2}\frac{x_1}{\sqrt{r}}e^{-kr}$. Then the full solution of the potential including the perturbation is

$$\Psi(r,\varphi,z) = C_0\left(\sqrt{r} - \frac{1}{2}\frac{x_1}{\sqrt{r}}e^{-kr}\cos(kz)\right)\cos(\varphi/2), \tag{3}$$

which in the vicinity of the perturbed edge can be presented as

$$\Psi|_{x=x_1} \approx \left.\left(\Psi_0 + \frac{\partial\Psi_0}{\partial x}x_1 + \varepsilon\Psi_1\right)\right|_{r\to0} = \tilde{C}_0\sqrt{r}\cos(\varphi/2),$$

where $\tilde{C}_0 = C_0\left(1 + \frac{1}{2}\hat{x}k\cos(kz)\right)$. Thus, the perturbation of the edge results in replacement of the constant C_0 by $\tilde{C}_0$ in the solution for the unperturbed solution. The magnetic flux and the current lines along the layer in the vicinity of the perturbed edge are shown in Fig. 2b.

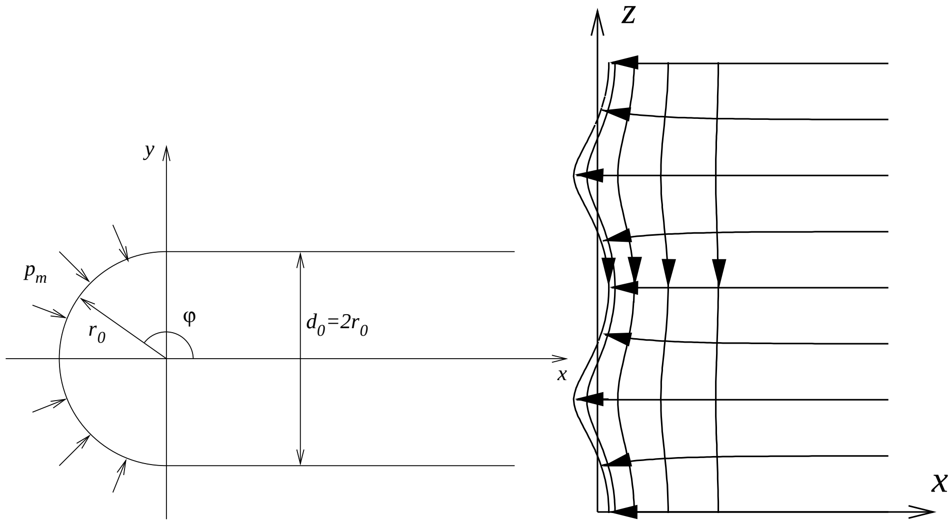

Fig. 2. (*a*) Evaluation of the magnetic pressure on the edge of the layer. (*b*) Magnetic flux and current lines along the layer in the vicinity of the perturbed edge.

In the perfect conductor approximation, the AC magnetic field generates an effective magnetic pressure on the surface of the layer with a time-averaged value $p_m = |\mathbf{B}|^2/4\mu_0$, where is $\mathbf{B}$ is the amplitude of the AC magnetic field at the sheet. As seen from solution (3), the magnetic pressure increases towards the edge as $\sim 1/r$ and, thus, it becomes singular at the edge. This singularity can be eliminated by considering a layer with a small but finite radius of curvature r_0. Then the magnetic pressure at the edge $\sim 1/r_0$ integrated over the "thickness" of the sheet $\sim r_0$ will result in a finite integral force independent of r_0. To evaluate the integral force on the edge, we assume $r_0 = d_0/2$ to be constant, as shown in Fig. 2*a*). Then the integration over the edge yields to up the first order terms

$$F = \lim_{r_0 \to 0} \int_{-r_0}^{r_0} p_m \mathrm{d}y = -\frac{1}{4\mu_0} \lim_{r_0 \to 0} r_0 \int_{\pi/2}^{3\pi/2} B^2(r_0, \varphi)\cos(\varphi)\mathrm{d}\varphi = F_0 + F_1,$$

where $F_0 = C_0^2/8\mu_0$ and $F_1 = F_0 \hat{x} k \cos(kz)$ are the base force and its perturbation, respectively. Further, we assume the sheet to be an inviscid liquid and consider a small-amplitude potential flow in the sheet caused by the perturbation of the edge. Thus, the linearised Euler equation applied to a potential velocity field $\mathbf{v} = \boldsymbol{\nabla}\Phi$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla} p = \nabla \left(\rho \frac{\partial \Phi}{\partial t} + p \right) = 0$$

leads to the pressure distribution in the sheet $p = p_0 - \rho \partial\Phi/\partial t = p_0 + p_1$, where p_0 is a constant base pressure, while $p_1 = -\rho\partial\Phi/\partial t$ is a perturbation of the pressure. The velocity potential Φ is governed by the incompressibility constraint $\nabla \cdot \mathbf{v} = 0$, which results in $\boldsymbol{\nabla}^2\Phi = 0$. At the free edge $x = 0$, we suppose the balance of normal stress averaged over the thickness of sheet: $\int_{-d_0/2}^{d_0/2} (p - \gamma/R)\,\mathrm{d}y \approx (p - \gamma/R)\,d_0 = F$ that results in $p|_{x=0} = \gamma/R + F/d_0$, where γ is the surface tension and $1/R$ denotes the curvature of the edge. For an unperturbed edge we have $p_0 = \gamma/R_0 + F_0/d_0$. Then, for the perturbation the balance condition takes the form

$$-\rho \left.\frac{\partial \Phi}{\partial t}\right|_{x=0} = \frac{\gamma}{R_1} + \frac{F_1}{d_0}, \tag{4}$$

where $1/R_1 \approx \nabla^2 x_1$ is the perturbation of the curvature of the edge. In addition, we have a kinematic constraint at the edge

$$v_x|_{x=0} = \left.\frac{\partial\Phi}{\partial x}\right|_{x=0} = \frac{\partial x_1}{\partial t}. \tag{5}$$

Henceforth, we search for the amplitude of the edge perturbation in the form $\hat{x}(t) = x_0 e^{\lambda t}$, where λ is, in general, a complex growth rate, whose real part has to be negative for the perturbation to be stable. In addition, all quantities are supposed to be constant over the thickness of the sheet because of smallness of the latter. Thus, searching the hydrodynamic potential as $\Phi(x,z,t) = \hat{\Phi}(x)\cos(kz)e^{\lambda t}$, we obtain the equation $\frac{d^2\hat{\Phi}}{dz^2} - k^2\hat{\Phi} = 0$, whose solution decaying away from the edge is $\hat{\Phi}(x) = \Phi_0 \mathrm{e}^{-kx}$. The amplitude of the hydrodynamic potential is related to that of the edge perturbation by the kinematic constraint (5) $\Phi_0 = -\frac{\lambda}{k}x_0$. Finally, the normal stress balance (4) yields $\lambda = k\sqrt{\frac{1}{\rho}\left(\frac{F_0}{d_0} - k\gamma\right)}$, which implies that long-wave perturbations with wavenumbers $k < k_c = \frac{F_0}{\gamma d_0}$ are unstable. Thus, the stronger the the linear electromagnetic force density on the edge F_0, the shorter the critical wavelength k_c. The waves, which are shorter than the critical, are stabilised by the surface tension. Although the long waves are always unstable, their growth rate reduces as $\sim k$ for $k \to 0$. Thus, there are perturbations with $k_{\max} = \frac{2}{3}k_c$, for which the growth rate has the maximum $\lambda_{\max} = k_{\max}\sqrt{\frac{1}{3\rho}\frac{F_0}{d_0}}$.

3. Conclusions We have obtained an analytical solution for the model of a semi-infinite sheet with a straight edge showing that the long-wave perturbations are unstable when the wavenumber exceeds some critical value k_c depending only on the surface tension and the density of electromagnetic force at the edge. The higher the density of the electromagnetic force, the shorter the critical wavelength. The perturbations with a wavelength shorter than the critical one are stabilised by the surface tension, whereas the growth rate of long-wave perturbations reduces as $\sim k$ for the wavenumbers $k \to 0$. Thus, there are the fastest growing perturbations with the wavenumber $k_{\max} = \frac{2}{3}k_c$.

REFERENCES

1. D. Perrier, Y. Fautrelle and J. Etay. Free surface deformations of a liquid metal drop submitted to a middle-frequency AC magnetic field. In *Proceedings of the 4th International Conference on Electromagnetic Processing of Materials* – EPM2003, (Lyon, France, 14–17 October 2003), C6-4.6. (on CD).

2. J.-U. Mohring and Ch. Karcher. Stability of a liquid metal interface affected by a high-frequency magnetic field. In *Proceedings of the 4th International Conference on Electromagnetic Processing of Materials* – EPM2003, (Lyon, France, 14–17 October 2003), A4-3.18. (on CD).

3. E. J. McHale and J. R. Melcher Instability of a planar liquid layer in an alternating magnetic field. *J. Fluid Mech.* vol. 114 (1982), pp. 276–40.

4. Y. Fautrelle and A. D. Sneyd Instability of a plane conducting free surface submitted to an alternating magnetic field. *J. Fluid Mech.* vol. 375 (1998), pp. 65–83.