TWO SHORT STORIES: THE U-SHAPED VELOCITY PROFILE AND THE ALIGNED JET

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Introduction. Recent results on both the formation of the U-shaped velocity profile at the entry in a uniform magnetic field (Alboussière, 2004, [1]; Andreev *et al.* , 2005, [2]) and the evolution of a round jet in the presence of an axial uniform magnetic field (Kim & Choi, 2004, [3]) have renewed the interest on these two problems, which are kinds of paradigms in liquid metal MHD. The purpose of this paper is to discuss some guiding ideas on the mechanisms which govern these flows and to derive relevant scaling laws. In the case of the entry problem (section 2 below), we consider the case of a rectangular channel, we suppose the initial formation of the U-shaped velocity profile in the fringing region is known and we take the Alboussière predictions as initial conditions at the beginning of the region where the magnetic field remains uniform. We focus on the evolution along the abscissa of this velocity distribution in the presence of the uniform magnetic field. This investigation is made in the frame of the quite usual assumptions for liquid metal MHD flows ($\mathrm{Re} \gg 1$, $\mathrm{Rm} \ll 1$, $\mathrm{Ha} \gg 1$, all walls being electrically insulating).

In the case of the aligned jet, we also assume $\mathrm{Re} \gg 1$ and $\mathrm{Rm} \ll 1$, and we use the simplified form of the induction equation valid in such conditions. No assumption is necessary on Ha, which is not a relevant parameter, but the minimum value of the magnetic field which is necessary to get a significant MHD effect is discussed, in terms of the Alfven number A/U, where A is the Alfven velocity and U the typical jet velocity, and of the magnetic Prandtl number $\mathrm{Pm} = \mu\sigma\nu$. The potential influence of Alfven waves on unsteady disturbances, which may develop because of the high shear present around the jet [5], is also discussed.

1. Evolution of the U-shaped velocity profile. Let us consider a rectangular channel of half-widths a in the magnetic field direction $(0z)$ and b in the direction $(0y)$ perpendicular to both the magnetic field and the duct axis $(0x)$. We start from the Alboussière [1] prediction on the jet formation, derived from a local analysis in the entry region where the applied uniform magnetic field is denoted $B_z(x) = B_0 B$ and where its axial derivative is written as

$$\frac{\mathrm{d}B_z}{\mathrm{d}x} = \frac{B_0}{a}\,G(x).$$

According to Alboussière [1], when the jet enters the uniform magnetic field ($B = 1$), the jet velocity profile has the exponential form

$$\frac{u}{u_1} = \exp\Big(-\frac{y}{a}\,G\sqrt{\mathrm{Ha}}\Big), \tag{1}$$

where $u_1(x)$ stands for the maximum of the jet over-velocity and the origin of y is taken at the wall. The Hartmann number $\mathrm{Ha} = aB_0\sqrt{\frac{\sigma}{\rho\nu}}$ is supposed very large. According to (1), the initial width of the jet is $\delta_0 = aG^{-1}\mathrm{Ha}^{-1/2}$ and G must be supposed small enough in comparison with unity, so that the side layer present along the parallel wall ($y = 0$), whose thickness is $\delta_\parallel = a\mathrm{Ha}^{-1/2}$, is significantly thinner than the jet and may be treated as a sub-layer (see Fig. 16 in Alboussière [1] to appreciate the validity of this assumption).

We now consider the downstream evolution of the jet when $x > 0$. Modeling the Hartmann friction with the linear term introduced by Sommeria & Moreau [4], the momentum equations in the axial direction within the jet and within the core flow, where the core velocity is denoted $u_0(x)$, write

$$u\frac{\partial u}{\partial x} + \nu\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} - \frac{u}{t_H} + \nu\frac{\partial^2 u}{\partial y^2}, \qquad u_0\frac{\mathrm{d}u_0}{\mathrm{d}x} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} - \frac{u_0}{t_H}. \tag{2)(3}$$

As soon as the pressure becomes uniform in the cross-section, the difference between (2) and (3) yields

$$u\frac{\partial u}{\partial x} + \nu\frac{\partial u}{\partial y} - u_0 = \frac{u_0 - u}{t_H} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{4}$$

where the two terms of the right hand side are of the order of $\frac{\nu U}{a^2}\mathrm{Ha}$, whereas the terms on the left hand side are of the order of U^2/L, U being a typical velocity scale and L a typical length in the x-direction. This implies that the jet's thickness cannot vary significantly over very long distances. We then admit that this thickness remains constant and equal to $\sqrt{\nu t_H} = a\mathrm{Ha}^{-1/2}$ all along the jet. This order of magnitude analysis also means that the key ingredient to accelerate the core flow is the pressure gradient occurring in the jet to balance the strong Hartmann braking force (of the order of $(u_0 + u_1)/t_H$) and also present in the core flow where the braking is only u_0/t_H. It also means that the transfer of momentum eventually due to the quasi-2D turbulence generated between the high velocity jet and the low velocity core, certainly related to the inertial terms of (4), cannot be a sufficient mechanism to accelerate the core flow.

Then, it is straightforward to derive two model equations for the unknown quantities $u_0(x)$ and $u_1(x)$ from the conservation of the total flow rate and the global momentum budget. In non-dimensional notations, the simplified form of these equations is:

$$U_1 = \alpha(1 - U_0) \qquad \text{and} \qquad \frac{\mathrm{d}U_0}{\mathrm{d}x}\big[(2-\alpha) + (\alpha - 3)U_0\big] = -(1 - U_0). \tag{5)(6}$$

Here U_0 and U_1 are non-dimensional velocities built with the velocity $V = \frac{q}{2a}$ of the equivalent uniform flow, x is the non-dimensional coordinate built with the length $Va^2/\nu\mathrm{Ha}$, and the parameter $\alpha = b/\delta = bG\sqrt{\mathrm{Ha}}/a$ is an aspect ratio characterizing the fraction of the total width occupied by the jet. The linear term at the right hand side of (6) represents the Hartmann braking supported by the jet. This simplified expression (6) is only valid when G is small enough, or when the Hartmann braking supported by the core flow is negligible in comparison with the braking supported by the jet. It could still be more simplified, assuming that $\alpha \gg 1$, but this would only be valid when $G\sqrt{\mathrm{Ha}} \gg a/b$. It is noticeable that, in the more general expression, this friction term would include both the friction in the Hartmann layer and the friction in the side layer, which are of same order of magnitude since the jet's thickness is proportional to $\mathrm{Ha}^{-1/2}$, whereas, in the simplified form (6), the friction along the side wall is neglected. Equations (5) and (6) have an explicit solution:

$$(1 - U_0)\exp\big[(3-\alpha)U_0\big] = \exp(-x). \tag{7}$$

At the entry into the magnet, the slope of the curve $U_0(x)$ is $(\alpha - 2)^{-1}$, which shows that the necessary length to reach a quasi-uniform velocity profile is of the order of $bG\sqrt{\mathrm{Ha}} - 2a$. And the asymptotic tendency to uniform the velocity distribution is given by the exponential expression:

$$U_0 = 1 - \exp(\alpha - 3 - x). \tag{8}$$

2. Main properties of the aligned jet. Let us write the actual magnetic field $\mathbf{B} = (B_0 + b_x)\mathbf{e_x} + \mathbf{b_r e_r}$, where b_x, b_r denote the axial and radial components of the induced magnetic field and B_0 stands for the intensity of the applied field. In the frame of the classical boundary layer approximation, the main novelty of this problem comes from the equation for the pressure variation in the radial direction (across the jet):

$$-\frac{1}{\rho}\frac{\partial}{\partial r}\Big(p+\frac{B_0 b_x}{\mu}\Big)=0\,, \tag{9}$$

which is raising the unusual question: is the pressure variation due to the radial component of the Lorentz force which tends to pinch the jet significant or not? To clarify this point, let us examine the expression of Ohm's law in the azimuthal direction (the two other components of the current density are zero or negligible):

$$\mu j_\theta=-\frac{\partial b_x}{\partial r}=-\mu\sigma B_0 \nu\,. \tag{10}$$

Assuming that those three terms are zero outside the jet (when $r\to\infty$) would cancel all fluid entrainment into the jet and would force the flow rate to be invariant, a situation we may consider as in contradiction with the definition of such a jet flow. Similarly, assuming that B_r cancels outside the jet would imply that the magnetic flux across any jet cross-section is invariant. So we reject those assumptions.

Now the motion equation in the axial direction writes:

$$u\frac{\partial u}{\partial x}+\nu\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}+\nu\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) \tag{11}$$

and the only way for the Lorentz force to intervene is through the axial component of the pressure gradient. Integrating (9) and using the condition that p and b_x both cancel outside the jet, we get $\frac{\mathrm{d}p}{\mathrm{d}x}=-\frac{B_0}{\mu}\frac{\partial b_x}{\partial x}=-\rho A\frac{\partial a}{\partial x}$, where $A=\frac{B_0}{\sqrt{\mu\rho}}$ stands for the Alfven velocity and $a=\frac{b_x}{\sqrt{\mu\rho}}$. To complete the formulation of this problem, we just have to add to (11) the equation of continuity and the projection of the induction equation in the axial direction:

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(r\nu)=0\,,\qquad A\frac{\partial u}{\partial x}+\eta\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial a}{\partial r}\Big)=0\,. \tag{12)(13}$$

And the boundary conditions are the same as for ordinary jets, with the addition of $a(r\to\infty)=\frac{\partial a}{\partial r}(r=0)=0$. Then the only novelty in this formulation comes from the new term $A\frac{\partial a}{\partial x}$ at the right hand side of equation (11).

Let us now introduce a typical velocity scale U (for instance the velocity at the exit of the nozzle). When compared with inertia, the new term is of the order of $\frac{A^2}{U^2}\frac{\nu}{\eta}$. Clearly, because the magnetic Prandtl number $\mathrm{Pm}=\frac{\nu}{\eta}=\mu\sigma\nu$ is always very small (10^{-7} to 10^{-6} in most liquid metals), a very large Alfven velocity is necessary to get a non negligible influence of the applied magnetic field on the jet. The enclosed table gives typical values of the ratio $\frac{A^2}{U^2}\frac{\nu}{\eta}$ for two values of U and three values of B_0. In all cases, it is clear that the laminar jet should be independent of the magnetic field influence.

	$B_0 = 1$ Tesla	$B_0 = 10$ Tesla	$B_0 = 10^2$ Tesla
$U = 1$ m/s	10^{-5}	10^{-3}	10^{-1}
$U = 3.16$ m/s	10^{-6}	10^{-4}	10^{-2}

Then, the main idea suggested by this analysis is that the effect of the magnetic field should be limited to the development of instabilities excited by the velocity gradient, certainly elongated in the magnetic field direction, and limited by both Joule and viscous dissipations. The numerical results presented by Kim & Choi [3] at the recent ICTAM (Warsaw, 2004) seem to agree fairly well with the above ideas. It is likely that Alfven waves, which are relevant at laboratory scale if the magnetic field is very high [5], or when the Lundquist number is larger than unity, may have a significant influence on these instabilities. In this jet flow, the relevant Lundquist number should be $\mathrm{Lu} = \dfrac{A\delta^2}{\eta x}$. Since the jet spreading is not significantly affected, its radius should be close to the classical laminar prediction $\delta \approx \dfrac{\nu x}{Ud}$, where U and d are typical velocity and length scales at the nozzle exit. This shows that the relevant Lundquist number is $\dfrac{A}{U}\dfrac{\nu}{\eta}\dfrac{\nu}{Ud}\dfrac{x}{d}$. In other words, even with a very low magnetic Prandtl number, there are always abscissas such that the waves become relevant. The consequence of this is that the simplest form of the induction equation should not be limited to (13), but should also keep the time derivative $\dfrac{\partial \mathbf{a}}{\partial t}$.

3. Concluding remarks. Simple ideas, based on the leading mechanisms, seem to be sufficient to predict the behavior of those two kinds of jet flows, at least in the asymptotic limit of high magnetic field. Remarkably, in both cases, the turbulence should not be an important ingredient. In the case of the U-shaped velocity profile, this analysis assumes that the duct aspect ratio is large enough to let enough space between the jets for a core flow (this is why we prefer the expression U-shaped to the usual M-shaped). It also assumes that the magnetic field gradient G is small enough. In a typical experiment G first slowly increases from zero when $x \to -\infty$ to a maximum of the order of 0.37 (this estimates comes from an exact solution for the curl-free and div-free magnetic field). It then decreases very rapidly to zero when $x \to 0$ and $B \to 1$. This suggests that the jet reaches its minimum width when $G = 0.37$ and then enters into the uniform field with this initial width $\delta_0 = 2.7a\sqrt{\mathrm{Ha}}$. This value of G is not very small and this ratio $\dfrac{\delta_0}{\delta_\parallel} = 2.7$ is not very large. One might therefore see here some limitation in the above results for this developing flow. In the case of the aligned jet, the above analysis suggests that the main effect of the applied magnetic field should be on the developments of instabilities and that the influence of the Alfven waves might be significant at a large distance from the nozzle.

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