LES MODELING OF ANISOTROPIC MHD TURBULENCE

O. Zikanov, A. Vorobev

Department of Mechanical Engineering, University of Michigan – Dearborn, Dearborn, MI 48128-1491, USA

1. Introduction and Numerical Method. Turbulent fluctuations in MHD flows become anisotropic under the action of a sufficiently strong magnetic field. As was deduced in [1], the mechanism of this transition is particularly transparent in the case of low magnetic Reynolds number $\mathrm{Rm} = uL/\eta$, which is considered in this paper. The rate of Joule dissipation of a Fourier mode with the wavenumber vector $\mathbf{k}$ is $\mu(\mathbf{k}) = \sigma B^2 \rho^{-1} (\widehat{\mathbf{v}}(\mathbf{k},t) \cdot \widehat{\mathbf{v}}^*(\mathbf{k},t)) \cos^2\theta$, where θ is the angle between $\mathbf{k}$ and the imposed magnetic field $\mathbf{B}$. The Joule dissipation is anisotropic, attaining the maximum for modes with $\mathbf{B} \parallel \mathbf{k}$ and zero for modes with $\mathbf{B} \perp \mathbf{k}$. The dissipation tends to eliminate the velocity gradients in the direction of $\mathbf{B}$ and elongate the flow structures in this direction. A remarkable feature of the flow transformation is that the relative rate of the dissipation $\mu(\mathbf{k})/\widehat{\mathbf{v}}^2$ depends on the angle θ but not on the wavenumber k. One can assume that the anisotropy would develop equally on all length scales of the flow. The situation looks more complicated if one takes into account the non-linear energy transfer between the modes and the resulting tendency to restoration of isotropy. The linearized picture of the flow development is, strictly speaking, correct only in the limit of infinitely large magnetic interaction parameter $N = \sigma B^2 L/\rho u$, when the inertia force is negligible in comparison with the Lorentz force. At finite N, one can expect a more complex scenario, probably with a scale-dependent anisotropy.

The flow transformation was studied in analytical, experimental, and numerical works, [1]–[7] among them. Some aspects, however, remains not fully understood. In particular, it is not clear, how universal is the anisotropy behaviour, i.e., to what degree it is determined by the magnetic interaction parameter as opposite to the length scale under consideration and specific features of a particular flow, such as large-scale forcing or the Reynolds number. Furthermore, there is a related question whether existing LES (Large Eddy Simulation) models can be justifiably applied to strongly anisotropic flows and which, if any, modifications of the models have to be made. In this paper, we try to answer the questions in a series of numerical experiments.

We consider a flow of a viscous, incompressible and electrically-conducting fluid in the presence of a constant uniform vertical magnetic field $\mathbf{B} = B\mathbf{e}_z$. The Lorentz force in quasi-static approximation is applied. Since our goal is to study the properties of turbulent fluctuations far from walls, the flow is assumed spatially homogeneous and calculated in a rectangular box $2\pi \times 2\pi \times 4\pi$ with periodic boundary conditions. An artificial forcing is applied to the Fourier modes with $1.5 \leq k \leq 3.1$ to generate a statistically steady flow. Two types of forcing mechanism are used. One is deliberately isotropic in the sense that the work is equally divided among the forced modes. To reveal the effect of the large-scale flow behaviour on anisotropy at smaller scales, we perform a series of simulations with a two-dimensional forcing, when only the modes with $k_z = 0$ are forced. This forcing imposes its own anisotropy at large scales. In order to investigate the effect of the Reynolds number and consider the anisotropy at larger scale separation than is possible in DNS, calculations at higher Re are carried out using the standard dynamic Smagorinsky LES model. The turbulent stresses are modeled using the eddy viscosity formula

$$\tau_{ij} - \delta_{ij}\tau_{kk} = -2C_{\mathrm{S}}\Delta^2 \, |S| \, S_{ij}, \qquad (1)$$

where $S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$, $\Delta = \sqrt[1/3]{\Delta_x \Delta_y \Delta_z}$ is the filter width, and $|S| = \sqrt{2S_{ij}S_{ji}}$. The Smagorinsky constant C_{S} is evaluated at each time step based on the assumption that (1) is universally valid at all inertial range length scales (see [8] and [9] for details). Applicability of traditional LES models to strongly anisotropic MHD turbulence is not obvious. In our case, however, the Reynolds number is not very high so only a small fraction of viscous dissipation has to be modeled. It is shown in [10] that the dynamic model is quite accurate in the simulations of homogeneous MHD turbulence at such Re. This conclusion is confirmed by our comparison between DNS and LES flows. At last, as we discuss below, our results indicate that the dynamic model, due to its self-adjusting mechanism, may be capable of adapting to anisotropic character of MHD turbulence. The details of the forcing and numerical algorithm can be found in [11].

2. Results and Discussion. In each numerical experiment, a developed non-magnetic turbulent flow is calculated, which is then used as an initial condition for three runs with $\mathrm{N}(t_0)$ equal to 0, 1, and 5. The DNS are performed with numerical resolution $256^2 \times 512$ and $\mathrm{Re}_\lambda(t_0) \approx 94$. For LES, we carried out two ‘test runs with the same parameters as DNS but using only $32^2 \times 64$ and $64^2 \times 128$ Fourier modes. A series of LES with $64^2 \times 128$ modes is performed with a gradually increasing Reynolds number, the highest value being $\mathrm{Re}_\lambda(t_0) = 290$. The LES run with $\mathrm{Re}_\lambda(t_0) = 150$ was done with 3D and 2D forcing.

The main results are presented in Fig. 1 and 2 (a more detailed account of our work can be found in [11] and in an extended paper currently in preparation). Fig. 1 shows the time-averaged curves of

$$g(k) \equiv \frac{3\tau}{2}\frac{\mu(k)}{E(k)} = \frac{3\sum \frac{k_z^2}{k^2}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}^*}{\sum \hat{\mathbf{u}}\cdot\hat{\mathbf{u}}^*}, \qquad (2)$$

used as a measure of dimensional anisotropy (the anisotropy of flow gradients) at the wavelength k. The sums are over all $\mathbf{k}$ in the shell $k - 1/2 < |\mathbf{k}| \leq k + 1/2$. Isotropic and purely two-dimensional flows correspond, respectively, to $g = 1$ and $g = 0$. An important conclusion that can be drawn from Fig. 1 is that there is a significant range of wavelengths, within which the dimensional anisotropy varies only slightly with k. This phenomenon was repeatedly observed in all our simulations, DNS and LES with different Reynolds numbers and numerical resolution. What is more, Fig. 1*d* shows that the flows obtained with 3D and 2D forcing have different anisotropy only at large scales, where it is directly introduced by the forcing. At smaller scales, the effect of large-scale dynamics quickly subsides and the values of $g(k)$ become very close. To conclude, our simulations strongly suggest that the dimensional anisotropy at intermediate and small length scales is a nearly universal function of the magnetic interaction parameter, affected only slightly by the scale, Reynolds number, and details of the large-scale dynamics.

We analyzed the anisotropy of velocity components at different length scales calculating the ratio

$$c(k) = (E_1(k) + E_2(k))/(2E_3(k)), \; E_i(k) = \sum_{k-\frac{1}{2}<|\mathbf{k}|\leq k+\frac{1}{2}} (\widehat{v}_i(\mathbf{k}) \cdot \widehat{v}_i^{\,*}(\mathbf{k}))/2. \qquad (3)$$

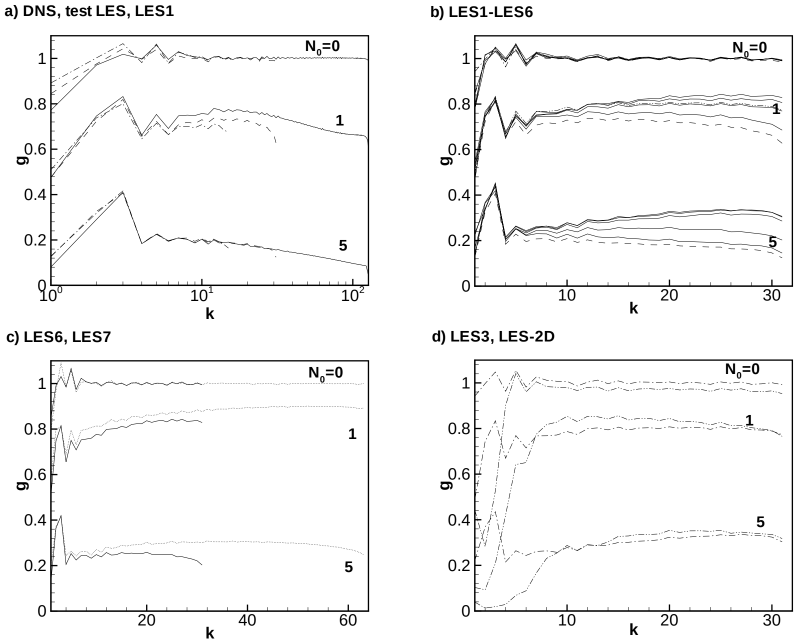

Fig. 1. Scale-dependent coefficient of dimensional anisotropy (2). (*a*) Comparison between the DNS and test LES calculations; (*b*) LES results obtained at $\mathrm{Re}_\lambda(t_0)$ ranging between 90 and 290; (*c*) LES results at $\mathrm{Re}_\lambda(t_0) = 290$ with numerical resolution $64^2 \times 128$ and $128^2 \times 256$; (*d*) LES results obtained with 3D (– – –) and 2D (– · · – · · –) forcing.

Form and existence of such an anisotropy are not obvious and do not follow directly from the action of the magnetic field. Still, we can see in Fig. 2 that the principal tendency is similar to that detected for the dimensional anisotropy. The coefficients

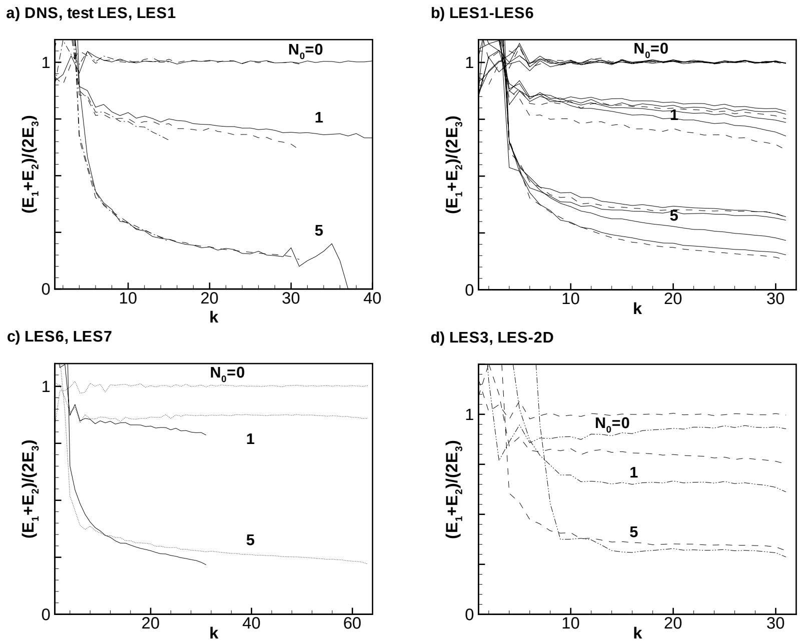

Fig. 2. Scale-dependent coefficient of anisotropy of velocity components (3). The notation is the same as in Fig. 1.

$c(k)$ at length scales sufficiently distanced from the forced region are much stronger affected by the value of N than by the scale, Reynolds number, and details of forcing.

The main focus of our current work is on evaluation of different strategies of LES modeling of strongly anisotropic MHD turbulence. All the considered models are based on the eddy viscosity hypothesis. The principal question is whether the Smagorinsky formula (1) can still be used, with the anisotropic character of the flow being represented by the anisotropy of S_{ij} and adjustment of C_S. There are indications (see [10] for decaying and [11] for forced flows at moderate Re_λ) that the dynamic model is not less accurate in the MHD case than it is for isotropic flows. This is not surprising in the view of our conclusion about the scale-invariance of anisotropy. The MHD correction, whatever it may be, is approximately the same at the length scales of grid and test filters. The mechanism of dynamic determination of C_S must, therefore, work in the MHD case with the same accuracy as in the non-magnetic isotropic flows.

Alternatively, one can modify (1) by introducing tensor-like eddy viscosity (see, e.g., [12] for development of such formula for the case of low-Rm MHD turbulence). Taking into account the remaining symmetries and adopting certain physics-based assumptions, one can write a generalization of (1) that includes 2 or 1 additional coefficients, for example

$$\tau_{ij} - \delta_{ij}\tau_{kk} = -2C_S\Delta^2\,|S| \begin{pmatrix} \kappa S_{11} - (1-\kappa)S_{33}/2 & 2\kappa S_{12} & S_{13} \\ 2\kappa S_{12} & \kappa S_{22} - (1-\kappa)S_{33}/2 & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix}.$$

Here κ is the anisotropy coefficient that can be evaluated dynamically or replaced by one of the integral characteristics of dimensional anisotropy, such as $G = \langle(\partial v_2/\partial z)^2\rangle/2\langle(\partial v_2/\partial y)^2\rangle$.

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